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Typed Quantum Logic

Kenji Tokuo[1]

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The aim of this paper was to lift traditional quantum logic to its higher order version with the help of a type-theoretic method. A higher order axiomatic system is defined explicitly and then a sound and complete class of models is given. This is an attempt to provide a quantum counterpart of classical "set theory" or intuitionistic "topos."

KEY WORDS: quantum logic; higher-order logic; type theory; set theory.

1. INTRODUCTION

Ordinary mathematics is carried out in classical set theory, which is based on classical logic. From this perspective, each choice of underlying logic must lead to a different set theory. A most noteworthy example of such a study is a construction of topos model for intuitionistic logic; an elementary topos can be considered as a universe of intuitionistic sets (Bell, 1988; Lambek, 1980). The following questions are then raised: Can we have a set theory based on *quantum logic*? If so, how is *quantum mathematics* different from ordinary mathematics?

Although several suggestions have been made to establish quantum set theory, none has been successful enough to be the basis of a rich mathematics. Takeuti (1981) transfinitely constructed his quantum set theory on the analogy of Boolean valued model for classical set theory. Unfortunately, however, it violates equality axioms by its very nature. One has to admit that no interesting mathematics could be founded without appropriate equality. Another approach is to extend the concept of quantum logic first, and then consider a set theory based on it. Nishimura's empirical set theory (Nishimura, 1995) was intended to be the foundation, not only of quantum mechanics, but of a more general setting—empirical sciences. His development of topos-like theory is quite fascinating; however, a subobject classifier and exponentials in the theory are allowed to exist only in very special cases. This seems to expose an intrinsic uncongeniality between empirical logic (in

[1] Graduate School of Human and Environmental Studies, Kyoto University, Kyoto, Japan; e-mail: tokuo@math.h.kyoto-u.ac.jp.

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particular, quantum logic) and topos theory. Finkelstein (1992) has also adopted a higher order language for quantum theory under the name of higher order quantum logic, employing the structure of Grassmann algebra; its style of formulation differs markedly from that of traditional lattice logic.

In this paper, we focus on *orthomodular lattice*—the most well-known representation of quantum logic (Dalla Chiara, 1984) and lift it in a very natural way to its higher order version with the help of a type-theoretic method. Section 2 provides a syntactic framework of our higher order quantum logic with equality, which is shown to be sufficiently expressive to develop a set theory. Section 3 presents a complete class of models for the syntax, that is, a universe of quantum sets. The consistency of the system is immediately derived from the existence proof of a model. Finally, some characteristic features of quantum sets are examined in Section 4.

2. FORMAL SYNTAX

2.1. Language

We set out to define our formal language $\mathcal{L}$ of order ω via a type-theoretic method, specifying *types* and *terms*.

Definition 2.1.1. (Types). The symbols for types of expressions are defined inductively as follows:

(i) 1 and Ω are types.
(ii) If necessary, one may have at most countably many additional symbols for types.
(iii) If A is a type, so is PA.
(iv) If A and B are types, so is $A \times B$.

Types which are not of the form PA or $A \times B$ are called *ground types*.

Definition 2.1.2. (Terms). Terms are expressions referring to objects or statements in a world. For each type A, a term of type PA is called a *set-like term*. A set-like term that contains no free variables is called an $\mathcal{L}$*-set*, or a quantum set, which is an expression of a property. A term of type Ω is sometimes called a *formula*, which is an expression of a fact. A formula that contains no free variables is called a closed formula or a *sentence*. The inductive definition is as follows:

(i) A symbol $*$ is a term of type 1. For each ground type A, one may add at most countable constants of type A to $\mathcal{L}$. A constant of type A is a term of type A.
(ii) For each type A, we give countably many variables x_A, y_A ... of type A, sometimes omitting the subscripts. A variable of type A is a term of type A.

(iii) If $\phi(x)$ is a formula that possibly contains a variable x of type A, then $\{x \in A \mid \phi(x)\}$ is a (set-like) term of type PA.
(iv) If a and b are terms of type A and B, respectively, then $\langle a, b\rangle$ is a term of type $A \times B$.
(v) If a is a term of type $A \times B$, then $(a)_1$ and $(a)_2$ are terms of type A and B, respectively.
(vi) If a and a' are both terms of the same type, then $a = a'$ is a formula.
(vii) If a and α are terms of type A and PA, respectively, then $a \in \alpha$ is a formula.
(viii) If p and q are both formulas, then so is $p \wedge q$.
(ix) If p is a formula, then so is $\neg p$.
(x) If $\phi(x)$ is a formula that possibly contains a variable x of type A, then $\forall x \in A.\phi(x)$ is a formula.

Remark. Parentheses are used to disambiguate expressions as usual. Some other useful symbols can be introduced as abbreviations:

- $p \vee q \equiv \neg(\neg p \wedge \neg q)$
- $p \Rightarrow q \equiv \neg p \vee (p \wedge q)$
- $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\exists x \in A.\phi(x) \equiv \neg(\forall x \in A.\neg\phi(x))$
- $\exists! x \in A.\phi(x) \equiv \exists x \in A.(\phi(x) \wedge \forall y \in A.(\phi(y) \Rightarrow (x = y))$
- $\top \equiv \neg(p \wedge \neg p)$
- $\perp \equiv \neg\top$
- $\{\langle x, y\rangle \in A \times B \mid \phi(x, y)\} \equiv \{z \in A \times B \mid \exists x \in A.(\exists y \in B.(z = \langle x, y\rangle \wedge \phi(z)))\}$

2.2. Rules

We now state the formal proof procedure for our higher order quantum logic. In the following, we write $A, B, \ldots$ for types, $p, q, \ldots$ for formulas, $a, b, \ldots$ for terms, and $x, y, \ldots$ for variables. $\phi(x)$ represents a formula that possibly contains a free variable x and $\phi(a)$ a formula obtained from $\phi(x)$ by replacing all free occurrences of x with a. The substitution is performed in the usual manner; technical details are omitted. For notational simplicity, we exclusively consider formulas with at most one free variable; the modification for multiple free variables is easy to perform. Γ denotes a finite (possibly empty) multiset of formulas, where a multiset means a set in which each element may occur more than once.

The intuitive meanings of the clauses listed below are as follows: The expression of the form $\Gamma \vdash p$ is called a *sequent*, asserting that one can syntactically deduce the formula p from the assumption of all the formulas in Γ. The expressions $\Gamma, p \vdash q$ and $p, q \vdash r$ mean $\Gamma \cup \{p\} \vdash q$ and $\{p, q\} \vdash r$, respectively. The

sequents above the horizontal line are the premises of the rule, and the one below it is the conclusion of the rule: if all the sequents above the line hold, so does the one below it. The one-line rules such as 2.1 are sometimes called *improper rules*, asserting that those sequents always hold without any premises.
Structural Rules

$$p \vdash p \tag{2.1}$$

$$\frac{\Gamma \vdash p \quad \Gamma, p \vdash, q}{\Gamma \vdash q} \quad \text{(Cut)} \tag{2.2}$$

$$\frac{\Gamma \vdash q}{\Gamma, p \vdash q} \tag{2.3}$$

$$\frac{\Gamma(x) \vdash \phi(x)}{\Gamma(a) \vdash \phi(a)} \tag{2.4}$$

Logical Rules

$$p \wedge q \vdash p \tag{2.5}$$

$$p \wedge q \vdash q \tag{2.6}$$

$$\frac{\Gamma \vdash p \quad \Gamma \vdash q}{\Gamma \vdash p \wedge q} \tag{2.7}$$

$$p \vdash \neg\neg p \tag{2.8}$$

$$\neg\neg p \vdash p \tag{2.9}$$

$$p \wedge \neg p \vdash q \tag{2.10}$$

$$\frac{p \vdash q}{\neg q \vdash \neg p} \tag{2.11}$$

$$\forall x \in A.\phi(x) \vdash \phi(a) \tag{2.12}$$

$$\frac{\Gamma \vdash \phi(x)}{\Gamma \vdash \forall x \in A.\phi(x)} \tag{2.13}$$

where x does not occur freely in Γ.

$$p \wedge \neg(p \wedge \neg(q \wedge p)) \vdash q \quad \text{(Orthomodular Law)} \tag{2.14}$$

Equality Rules

$$\vdash a = a \tag{2.15}$$

$$a = b, \phi(a) \vdash \phi(b) \tag{2.16}$$

$$\frac{p \vdash q \quad q \vdash p}{\vdash p = q} \tag{2.17}$$

Set Rules

$$\vdash z = * \tag{2.18}$$

where z is a variable of type 1.

$$\vdash ((\langle a, b\rangle)_1 = a) \wedge ((\langle a, b\rangle)_2 = b) \tag{2.19}$$

$$\vdash \langle (a)_1, (a)_2\rangle = a \tag{2.20}$$

$$\vdash (x \in \{x \in A \mid \phi(x)\}) = \phi(x) \qquad \text{(Comprehension)} \tag{2.21}$$

Remark. The reader may observe that our framework can be regarded as a natural higher order extension of traditional quantum logic. The structural and the logical rules essentially correspond to the axiomatic systems for propositional or first-order quantum logic, studied by Goldblatt (1974), Dalla Chiara (1984), and Nishimura (1980).

Formal Proof. *A diagram of rules that satisfies the following inductive specifications is said to be a* proof diagram. *The bottommost sequent of a proof diagram is called its* end sequent.

- (i) *An improper rule is itself a proof diagram.*
- (ii) *If P is a proof diagram whose end sequent is S, and S/T is one of the rules listed above, then P/T is a proof diagram whose end sequent is T.*
- (iii) *If P_1 and P_2 are both proof diagrams whose end sequents are S_1 and S_2, respectively, and $S_1\ S_2/T$ is one of the rules listed above, then $P_1\ P_2/T$ is a proof diagram whose end sequent is T.*

We say that the sequent T is provable *if and only if there exists a proof diagram whose end sequent is T.*

2.2.1. Derivable Rules

We next present some useful rules to shorten proofs, which are derivable from those shown in the list above. It turns out that our axiomatic system has sufficient expressive power to produce plenty of mathematically interesting theorems. We only provide sketches of some proofs or omit them entirely.

$$\vdash \top \tag{2.22}$$

We have $\neg p \wedge p \vdash \neg(a = a)$ by 2.10. Then $a = a \vdash \neg(\neg p \wedge p)$ by 2.11. Since $\vdash a = a$ by 2.15, we get $\vdash \neg(\neg p \wedge p)$ by 2.2.

$$\frac{\Gamma, p \vdash r}{\Gamma, p \wedge q \vdash r} \tag{2.23}$$

$$\frac{\Gamma, q \vdash r}{\Gamma, p \wedge q \vdash r} \tag{2.24}$$

$$\frac{p \vdash r \quad q \vdash r}{p \vee q \vdash r} \tag{2.25}$$

$$p \vdash p \vee q \tag{2.26}$$

$$q \vdash p \vee q \tag{2.27}$$

$$p = q \vdash p \Leftrightarrow q \tag{2.28}$$

$$\frac{p \vdash q}{\vdash p \Rightarrow q} \quad \text{(Deduction Theorem)} \tag{2.29}$$

We have $p \vdash q$ by assumption and $p \vdash q$ by 2.1. Then we obtain $p \vdash p \wedge q$ by 2.7. We also have $p \wedge q \vdash p$ by 2.5. Hence we infer $\vdash p = p \wedge q$ by 2.17. Applying 2.2 to this and 2.28 establishes that $\vdash p \Leftrightarrow p \wedge q$. In particular, $\vdash p \Rightarrow p \wedge q$. We get the conclusion by using the provable sequent $\vdash (p \Rightarrow p \wedge q) = (p \Rightarrow q)$.

$$\frac{\Gamma \vdash p \Rightarrow q}{\Gamma, p \vdash q} \tag{2.30}$$

$$\frac{\phi(x) \vdash p}{\exists x \in A.\phi(x) \vdash p} \tag{2.31}$$

where x does not occur freely in p.

$$\phi(x) \vdash \exists x \in A.\phi(x) \tag{2.32}$$

$$a = a' \vdash a' = a \tag{2.33}$$

$$a = a', a' = a'' \vdash a = a'' \tag{2.34}$$

$$\langle a, b\rangle = \langle c, d\rangle \vdash (a = c) \wedge (b = d) \tag{2.35}$$

We have $\langle a, b\rangle = \langle c, d\rangle, (\langle a, b\rangle)_1 = (\langle a, b\rangle)_1 \vdash (\langle a, b\rangle)_1 = (\langle c, d\rangle)_1$ by 2.16. Hence we obtain $\langle a, b\rangle = \langle c, d\rangle \vdash a = c$ by 2.19. Similarly, $\langle a, b\rangle = \langle c, d\rangle \vdash b = d$. We then get the conclusion by 2.7.

$$(a = c) \wedge (b = d) \vdash \langle a, b\rangle = \langle c, d\rangle \tag{2.36}$$

$$\phi(x) = (x \in \alpha) \vdash \{x \in A \mid \phi(x)\} = \alpha \tag{2.37}$$

Let $\alpha \equiv \{x \in A \mid \psi(x)\}$. We have $\phi(x) = (x \in \{x \in A \mid \psi(x)\}) \vdash \phi(x) = \psi(x)$ by 2.21. We also have $\phi(x) = \psi(x), \{x \in A \mid \phi(x)\} = \{x \in A \mid \phi(x)\} \vdash \{x \in A \mid \phi(x)\} = \{x \in A \mid \psi(x)\}$ by 2.16. Applying 2.2 leads to the desired conclusion.

$$(x \in \alpha) = (x \in \beta) \vdash \alpha = \beta \quad \text{(Extensionality)} \tag{2.38}$$

We have $(x \in \alpha) = (x \in \beta) \vdash \{x \in A \mid (x \in \beta)\} = \alpha$ by letting $\phi(x) \equiv (x \in \beta)$ in 2.37. Then it suffices to show that $\{x \in A \mid (x \in \beta)\} = \beta$. We obtain

$(x \in \beta) = (x \in \beta) \vdash \{x \in A \mid (x \in \beta)\} = \beta$ by using 2.37 again.

$$\phi(x) = \psi(x) \vdash \{x \in A \mid \phi(x)\} = \{x \in A \mid \psi(x)\} \tag{2.39}$$

We have $\phi(x) = \psi(x) \vdash \phi(x) = (x \in \{x \in A \mid \psi(x)\})$ by 2.21. Applying 2.2 to this and 2.37 establishes the conclusion.

$$\{x \in A \mid \phi(x)\} = \{x \in A \mid \psi(x)\} \vdash \phi(x) = \psi(x) \tag{2.40}$$

We have $\{x \in A \mid \phi(x)\} = \{x \in A \mid \psi(x)\}, (x \in \{x \in A \mid \phi(x)\}) = (x \in \{x \in A \mid \phi(x)\}) \vdash (x \in \{x \in A \mid \phi(x)\}) = (x \in \{x \in A \mid \psi(x)\})$ by 2.16. We then get the conclusion by 2.21.

3. FORMAL SEMANTICS

This section provides mathematically strict meanings of the expressions in $\mathcal{L}$. For this purpose, a class of models for higher order quantum calculus is specified, roughly analogous to what is called Henkin models (Henkin, 1950) developed originally for classical higher order logic. First, the precise concept of validity is defined with respect to this class of models. Some important results are stated: Any provable sequents are valid with respect to this class of models (soundness), and conversely, any valid sequents with respect to this class of models are provable (completeness). In the completeness proof, a concrete general model, called the canonical model, is constructed. The existence of the model immediately implies the consistency of our axiomatic system.

3.1. General Model for $\mathcal{L}$

An $\mathcal{L}$-frame is defined to be a collection of nonempty sets $\{D_A\}_A (\equiv \{D_A, D_B, \ldots\})$ that satisfies the following conditions.

We associate with each ground type A a set D_A, which is referred to as the *domain* of type A. D_1 and D_Ω must in particular be a singleton set and an orthomodular lattice, respectively. An orthomodular lattice here means an orthocomplemented lattice satisfying the orthomodular law (Dalla Chiara, 1984). Its order, equality, inf, orthocomplementation, top, and bottom are denoted by $\leq, =, \wedge, *, \top$, and $\perp$, respectively (some symbols are the same as our logical symbols; this is not expected to cause confusion since it is clear from the context). The domains of the other types are constructed inductively. D_{PA} is a certain collection of subsets of D_A, i.e., $D_{PA} \subseteq 2^{D_A}$ and $D_{A\times B}$ is the usual set-theoretic product $D_A \times D_B$.

Remark. The reduced case where $D_\Omega = \{\top, \perp\}$ corresponds to a gerneral model for classical higher order logic (Andrews, 1972; Henkin, 1950). Another limit case where D_{PA} is taken to be the collection of *all* subsets of D_A corresponds to the Henkin's standard model (Henkin, 1950).

Definition 3.1.1 (Assignment). Given an $\mathcal{L}$-frame $\{D_A\}_A$, we define an *assignment* ρ as a map on the sets of variables in $\mathcal{L}$, satisfying the condition that $\rho(x_A) \in D_A$ for each type A. Given an assignment ρ, a variable x_A, and an element $d \in D_A$, we write $(\rho{:}x_A/d)$ for the assignment which is the same map as ρ except that the value of x_A is d.

Definition 3.1.2 (General Model). An $\mathcal{L}$-frame $\{D_A\}_A$ is said to be a *general model* $\mathcal{M}$ for $\mathcal{L}$ if there exists a map $[\cdot]_\rho$ on the set of all terms such that $[a]_\rho \in D_A$ for each term a of type A and for each assignment ρ, satisfying the following conditions:

(i) $[x_A]_\rho = \rho(x_A)$
(ii) $[\{x \in A \mid \phi(x)\}]_\rho = \{d \in D_A \mid [\phi(x)]_{(\rho:x/d)} = \top\}$
(iii) $[\langle a, b\rangle]_\rho = ([a]_\rho, [b]_\rho)$
(iv) $[(\langle a, b\rangle)_1]_\rho = [a]_\rho$
(v) $[(\langle a, b\rangle)_2]_\rho = [b]_\rho$
(vi) $[a = a']_\rho = \top$ if $[a]_\rho = [a']_\rho$
(vii) $[a = a']_\rho \wedge [\phi(a)]_\rho \leq [\phi(a')]_\rho$
(viii) $[a \in \alpha]_\rho = \top$ if $[a]_\rho \in [\alpha]_\rho$
(ix) $[a \in \{x \in A \mid \phi(x)\}]_\rho = [\phi(a)]_\rho$
(x) $[p \wedge q]_\rho = [p]_\rho \wedge [q]_\rho$
(xi) $[\neg p]_\rho = [p]_\rho^*$
(xii) $[\forall x \in A.\phi(x)]_\rho = \wedge_{d\in D_A}\{[\phi(x)]_{(\rho:x/d)}\}$

Definition 3.1.3 (Validity). Let $\mathcal{M}$ be a general model for $\mathcal{L}$. For $\Gamma \equiv \{p_1, p_2, \ldots, p_m\}$, we define $[\Gamma]_\rho$ as $[p_1]_\rho \wedge [p_2]_\rho \wedge \cdots \wedge [p_m]_\rho$ if $m > 0$; $\top$ otherwise. We say that $\Gamma \vdash p$ is *valid* in $\mathcal{M}$, or symbolically we write $\Gamma \models_{\mathcal{M}} p$, provided $[\Gamma]_\rho \leq [p]_\rho$ for any ρ. We write $\Gamma \models p$ if $\Gamma \models_{\mathcal{M}} p$ in every general model $\mathcal{M}$.

It may not be immediately clear whether there actually exists a general model that satisfies all the above conditions. In the next subsection, we explicitly construct a general model, each domain D_A of which is a countable set.

3.2. Soundness, Completeness, and Consistency

3.2.1. Soundness

The soundness theorem is stated as follows: If $\Gamma \vdash p$ is provable, then $\Gamma \models p$. We can show that the endsequent in every proof diagram is valid in every general model, by induction on the construction of the proof: Given any model, the improper rule is obviously turned out to be valid. For the induction step, one may verify that for each rule, if all the premises are valid, then the conclusion is also valid. Thus the proof of the soundness theorem is completed.

3.2.2. Completeness

The completeness theorem is stated as follows: If $\Gamma \models p$, then $\Gamma \vdash p$ is provable.

The proof begins by introducing the Lindenbaum algebra of $\mathcal{L}$. Let us define a relation $\sim$ on the set of all closed terms: $a \sim a'$ if and only if the sequent $\vdash a = a'$ is provable. We see that this relation is an equivalence relation: a relation that is reflexive, symmetric, and transitive. The equivalence class of a is denoted by $[a]_{\mathcal{L}}$. The set of all equivalence classes of *sentences* forms an orthomodular lattice with the following ordering relation and operations:

- $[p]_{\mathcal{L}} \leq [q]_{\mathcal{L}}$ iff $p \vdash q$ is provable.
- $\wedge([p]_{\mathcal{L}}, [q]_{\mathcal{L}}) \equiv [p \wedge q]_{\mathcal{L}}$
- $[p]^*_{\mathcal{L}} \equiv [\neg p]_{\mathcal{L}}$

This orthomodular lattice is denoted by L_0. In general, any orthomodular lattice is embeddable into some complete orthomodular lattice (Bell, 1983). A complete orthomodular lattice that contains L_0 as a suborthomodular lattice is referred to as L, to be used in the construction of the canonical model $\mathcal{M}_{\mathrm{c}}$ for $\mathcal{L}$. We give $\mathcal{M}_{\mathrm{c}}$ as a model for $\mathcal{L}'$, which is an extension of the language $\mathcal{L}$, by adding some constants. For us, it is sufficient to discuss $\mathcal{M}_{\mathrm{c}}$ since a model for $\mathcal{L}'$ is also a model for $\mathcal{L}$. So we abuse notation and simply write $\mathcal{L}$ for $\mathcal{L}'$.

We define the $\mathcal{L}$-frame $\{D_A\}$ of $\mathcal{M}_{\mathrm{c}}$ by induction on the construction of type A, with the one-to-one map Φ on the set of equivalence classes of closed terms, such that $\Phi([a]_{\mathcal{L}}) \in D_A$ for each term a of type A. For each ground type A, D_A is a set of all equivalence classes of closed terms of type A and Φ is the identity map. In particular, D_1 is a singleton set whose unique element is $[*]$, and D_Ω is L defined above. We extend the language $\mathcal{L}$ by adding for each $d \in D_\Omega$ the constant symbol $\bar{d}$ such that $\Phi(\bar{d}) = d$. For each the set-like term α of type *PA*, we first introduce the concept of property of α: Recall that α is of the form $\{x \in A \mid \phi(x)\}$ by definition. The property of α is then meant to be the equivalence class of $\phi(x)$, that is, $[\phi(x)]_{\mathcal{L}}$. When $\phi(x)$ is of the form $x \in \{x \in A \mid \phi'(x)\}$, we can also say $[\phi'(x)]_{\mathcal{L}}$ is the property of α, since we see that the sequent $\vdash \phi(x) = \phi'(x)$ is provable. Now assume that the domain D_A, and the map Φ from the set of all closed terms of type A onto D_A, have already been defined. Let $\Phi(\alpha)$ be the set $\{d \in D_A \mid \Phi([\phi(\bar{d})]_{\mathcal{L}}) = \top\}$, where $\bar{d}$ is a closed term of type A such that $\Phi(\bar{d}) = d$, and $[\phi(x)]_{\mathcal{L}}$ is the property of α. Then D_{PA}, is defined to be the totality of $\Phi(\alpha)$ of each set-like term α of type *PA*. Finally, $D_{A\times B}$ is constructed as the usual direct product $D_A \times D_B$ of the sets with the trivial map Φ.

Having so defined the $\mathcal{L}$-frame $\{D_A\}_A$ with the map $[a]_\rho \equiv \Phi([a_\rho]_{\mathcal{L}})$, we call it the *canonical model* $\mathcal{M}_{\mathrm{c}}$ for $\mathcal{L}$. Here a_ρ means the closed term obtained from a by replacing all free occurrences of x with the closed term $\bar{d}$ such that $\rho(x) = \Phi(\bar{d})$ D_{PA}.

The completeness proof using the canonical model $\mathcal{M}_c$ goes as follows: Suppose $\Gamma \models p$. In particular, $\Gamma \models_{\mathcal{M}_c} p$ for the canonical model $\mathcal{M}_c$. This means that $\Gamma \vdash p$ is provable. (For sequents with a free variable x, $\Gamma \models_{\mathcal{M}_c} p$ implies $\models_{\mathcal{M}_c} \Gamma \Rightarrow p$, that is, $\models_{\mathcal{M}_c} \forall x \in A.(\Gamma \Rightarrow p)$. Hence $\vdash \forall x \in A.(\Gamma \Rightarrow p)$ is provable, and so $\vdash \Gamma \Rightarrow p$ is provable, that is, $\Gamma \vdash p$ is provable.)

3.2.3. Consistency

A collection Γ of sentences is called *inconsistent* if the sequent $\Gamma \vdash \bot$ is provable; otherwise *consistent*.

Our higher order quantum logic $\mathcal{L}$ is consistent. We see this as follows: Suppose for contradiction that $\vdash \bot$ is provable. Then by the soundness theorem, $[\bot]_\rho$ is interpreted as $\top$ in every general model. This does not hold, however, in a model with D_Ω such that $\top \neq \bot$.

4. QUANTUM SET THEORY

4.1. Notation

Let us recall that a term of type PA for some type A is called a set-like term in $\mathcal{L}$ and a closed set-like term is called an $\mathcal{L}$-set. Note that it makes sense to write $x \in \alpha$ where x is a variable of some type A and α is an $\mathcal{L}$-set of type PA. We use the following notational conventions:

- $\forall x \in \alpha.\phi(x) \equiv \forall x \in A.((x \in \alpha) \Rightarrow \phi(x))$
- $\exists x \in \alpha.\phi(x) \equiv \exists x \in A.((x \in \alpha) \wedge \phi(x))$
- $\exists! x \in \alpha.\phi(x) \equiv \exists! x \in A.((x \in \alpha) \wedge \phi(x))$
- $\{x \in \alpha \mid \phi(x)\} \equiv \{x \in A \mid (x \in \alpha) \wedge \phi(x)\}$

The usual set-theoretic operations and relations are then defined as follows:

- $\alpha \subseteq \beta \equiv \forall x \in \alpha.x \in \beta$ (where α and β are of the same type PA.)
- $\alpha \cap \beta \equiv \{x \in A \mid (x \in \alpha) \wedge (x \in \beta)\}$ (where α and β are of the same type PA.)
- $\alpha \cup \beta \equiv \{x \in A \mid (x \in \alpha) \vee (x \in \beta)\}$ (where α and β are of the same type PA.)
- $\cap U \equiv \{x \in A \mid \forall \alpha \in U.x \in \alpha\}$
- $\cup U \equiv \{\in A \mid \exists \alpha \in U.x \in \alpha\}$
- U_A or $A \equiv \{x \in A \mid \top\}$
- ϕ_A or $\phi \equiv \{x \in A \mid \bot\}$
- $-\alpha \equiv \{x \in A \mid \neg(x \in \alpha)\}$
- $P\alpha \equiv \{u \in PA \mid u \subseteq \alpha\}$
- $\alpha \times \beta \equiv \{\langle x, y\rangle \in A \times B \mid (x \in \alpha) \wedge (y \in \beta)\}$ (where α is of type PA and β is of type PB. Both may be of the same type.)

- $\beta^\alpha \equiv \{u \in P(A \times B) \mid (u \subseteq (\alpha \times \beta)) \wedge \forall x \in \alpha.(\exists! y \in \beta.\langle x, y\rangle \in u)\}$. (where α is of type PA and β is of type PB. Both may be of the same type.)
- $\{a\} \equiv \{x \in A \mid x = a\}$

Remark. Some operations must obey the constraints on types shown in the parentheses. This is what Bell (1988) calls a "local" property.

4.2. Theorems on Sets

4.2.1. Provable Sequents on Sets

For example, the following sequents are derivable in $\mathcal{L}$: Our quantum set theory shares many of its characteristics with classical set theory.

$$\vdash \alpha = \beta \Leftrightarrow \forall x \in A(x \in \alpha \Leftrightarrow x \in \beta) \tag{4.1}$$

$$\vdash \alpha \subseteq \alpha \tag{4.2}$$

$$\vdash (\alpha \subseteq \beta \wedge \beta \subseteq \alpha) \Rightarrow \alpha = \beta \tag{4.3}$$

$$\vdash x \in U_A \tag{4.4}$$

provided x is of type A

$$\vdash \neg(x \in \phi_A) \tag{4.5}$$

provided x is of type A

$$\vdash \alpha \in P\beta \Leftrightarrow \alpha \subseteq \beta \tag{4.6}$$

4.2.2. Unprovable Sequents on Sets

Some familiar sequents are, however, unprovable in $\mathcal{L}$. These "weaker" features of quantum set theory are in sharp contrast with those of classical set theory. Specifically, we can construct countermodels for the following sequents:

$$(\alpha \subseteq \beta) \wedge (\beta \subseteq \gamma) \vdash \alpha \subseteq \gamma \tag{4.7}$$

This sequent expresses the law of transitivity. Suppose for contradiction that it is provable. Then it is easy to see that $(x \in \alpha \Rightarrow x \in \beta) \wedge (x \in \beta \Rightarrow x \in \gamma) \vdash x \in \alpha \Rightarrow x \in \gamma$ is also provable. Letting $p \equiv (x \in \alpha)$, $q \equiv (x \in \beta)$, and $r \equiv (x \in \gamma)$, we deduce $(\neg p \vee (q \wedge p)) \wedge (\neg q \vee (r \wedge q)) \vdash \neg p \vee (r \wedge p)$, which is not valid in the following model: Let D_Ω be the collection of all closed subspaces of $\mathbf{R}^3$ (3-dimensional real Euclidean space) and set $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (1, 1, 0)$, $[p]_\rho \equiv \mathbf{Span}(\vec{v}_1)$, $[q]_\rho \equiv \mathbf{Span}(\vec{v}_1, \vec{v}_2)$, and $[r]_\rho \equiv \mathbf{Span}(\vec{v}_2)$, where $\mathbf{Span}(\vec{v}_1, \ldots, \vec{v}_k)$ denotes the closed subspace spanned by the vectors $\vec{v}_1, \ldots, \vec{v}_k$. Thus we conclude by the soundness theorem that 4.7 is unprovable.

$$(\gamma \subseteq \alpha) \wedge (\gamma \subseteq \beta) \vdash \gamma \subseteq (\alpha \cap \beta) \tag{4.8}$$

Suppose for contradiction that this sequent is provable. This means that $(x \in \gamma \Rightarrow x \in \alpha) \wedge (x \in \gamma \Rightarrow x \in \beta) \vdash x \in \gamma \Rightarrow (x \in \alpha \wedge x \in \beta)$ is provable. Letting $p \equiv (x \in \alpha)$, $q \equiv (x \in \beta)$, and $r \equiv (x \in \gamma)$, we deduce $(\neg r \vee (p \wedge r)) \wedge (\neg r \vee (q \wedge r)) \vdash \neg r \vee (p \wedge q \wedge r)$. Further, letting $p' \equiv (p \wedge r)$, $q' \equiv (q \wedge r)$, and $r' \equiv (\neg r)$, we may rewrite this as $(r' \vee p') \wedge (r' \vee q') \vdash r' \vee (p' \wedge q')$. This is the very instance of the distributive law, which is abandoned in quantum logic. Therefore it is not valid in some models, and hence is unprovable in general.

$$(\alpha \cup \beta) \subseteq \gamma \vdash (\alpha \subseteq \gamma) \wedge (\beta \subseteq \gamma) \tag{4.9}$$

Note that $\vdash \alpha \subseteq (\alpha \cup \beta)$ is a provable sequent. Replacing β in 4.7 with $\alpha \cup \beta$ we may apply the discussion of the preceding paragraph.

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